Research Statement

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1 Introduction

My research is in the field of arithmetic geometry. Broadly, this is the area of mathematics that uses techniques from algebraic geometry to study problems and objects in number theory. Specifically, I am interested in studying algebraic curves and abelian varieties in positive characteristic and the general question of which abelian varieties can be realized as jacobians of algebraic curves. The essence of my research agenda is captured by the following question schema:

If an abelian variety has a certain property, is there a smooth curve whose jacobian has the same property?

In Section 2, we motivate the problems we study with some background. Next, we describe three specific results in sections 3, 4 and 5 respectively. Briefly, the three topics are:

1. Deciding when an isogeny class of abelian varieties over $\mathbb{F}_q$ does not contain the jacobian of a hyperelliptic curve.
2. Using the existence of curves with special Newton polygons to give lower bounds on the number of Dirichlet characters over vanishing at $s = \frac{1}{2}$ in the function field setting.
3. Explicitly computing discrete invariants of the p-torsion of the jacobians of families of curves when the associated family of jacobians is dense in an irreducible component of a Shimura variety.

2 Motivation

The theory of algebraic curves over finite fields has been a focus of intense research, not only as a striking parallel to the classical theory of algebraic number fields but also in it's own right as a subfield of algebraic geometry. One key point in which the two theories diverge is the existence of the Frobenius map and the ensuing non-separable extensions which are entirely absent in the characteristic 0 setting. As a result, some new tools exploiting this extra structure available in positive characteristic can make analogs of classical results easier to prove. Another tool providing much utility in studying an algebraic curve is it's jacobian variety, an abelian variety that encodes a lot of information about the curve. The interplay between curves and abelian varieties has been the subject of much fruitful research over the past century. My research focuses on this interplay in positive characteristic.

Given a curve C over a finite field $\mathbb{F}_q$, we can attach several types of discrete invariants to them. Some of these invariants, like the Newton polygon and Ekedahl-Oort type (defined in section 5) attached to their jacobians, are unchanged even after a base change to $\overline{\mathbb{F}}_q$. We term these as *geometric invariants*. Some other invariants like the $\mathbb{F}_q$-isogeny class are extremely sensitive to base-change. We call these *arithmetic invariants*. Even though two curves defined over $\mathbb{F}_q$ may not be isogenous over $\mathbb{F}_q$, they may become isogenous over a finite base-change to $\mathbb{F}_{q^d}$. For instance, any two supersingular curves of genus g over a finite field of characterstic p become isogenous over $\overline{\mathbb{F}}_q$.

In the first two of our applications, we study arithmetic invariants of jacobians, the $\mathbb{F}_q$-isogeny class and the binary invariant detecting whether $q^{\frac{1}{2}}$ is a Frobenius eigenvalue. In the last of our applications, we define and study geometric invariants, the Ekedahl-Oort type and the Newton polygon.

3 Hyperelliptic Jacobians in Isogeny Classes

Let A be an abelian variety of dimension g defined over a finite field $\mathbb{F}_q$. The $\mathbb{F}_q$-isogeny class of such an abelian variety A is uniquely determined by the characteristic polynomial of its Frobenius endomorphism. This polynomial is called the *Weil polynomial* of the abelian variety and has the following form:

$$Z_A(t) = t^{2g} + a_1 t^{2g-1} + \cdots + a_{g-1}t^{g+1} + a_g t^g + a_{g-1}qt^{g-1} + \cdots + a_1 q^{g-1} t + q^g \in \mathbb{Z}[t].$$

Given $Z_A(t)$, we ask if the isogeny class of A contains the jacobian of a smooth curve over $\mathbb{F}_q$.

In genus one, the problem is straightforward, as every one-dimensional abelian variety is an elliptic curve, which is isomorphic to its jacobian. For genus two, the problem was solved by Howe, Nart, and Ritzenthaler [6] who give an explicit classification using only elementary restrictions involving the integers a_1, a_2, and q. Their method crucially uses that every genus 2 curve is hyperelliptic and thus has a canonical involution.

Solving this problem in higher genus appears to be significantly more complicated. For example, every curve of genus three is isomorphic to either a hyperelliptic curve or to a smooth plane quartic curve. Curves of the latter type generically do not possess any non-trivial automorphisms, so the arguments of Howe, Nart and Ritzenthaler for genus two cannot easily be extended to non-hyperelliptic genus three curves.

In joint work [2] with Costa, Fernando, Karemaker, Springer and West, we focus on a more accessible question in arbitrary dimension: Does a given isogeny class of abelian varieties contain the jacobian of a hyperelliptic curve? We do this by analyzing the various possible geometric configurations of the Weierstrass points of hyperelliptic curves and their orbits under the action of Frobenius. In doing so, we obtain parity conditions on the coefficients of the Weil polynomials which prevent certain isogeny classes from containing the jacobian of a hyperelliptic curve. For instance, in genus three we obtain:

Theorem 3.1 ([2], Theorem 2.8)**.** *Let q be an odd prime power. The isogeny classes of three-dimensional abelian varieties corresponding to Weil polynomials of the form*

$$t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + q a_2 t^2 + q^2 a_1 t + q^3$$

with $a_2 \equiv 0 \pmod 2$ and $a_3 \equiv 1 \pmod 2$ do not contain the jacobian of a hyperelliptic curve over $\mathbb{F}_q$.

Furthermore, we demonstrate that the Weil polynomial of the jacobian of a hyperelliptic curve of genus g over a finite field of odd characteristic must be congruent modulo 2 to a polynomial of the form

$$\prod_{i=1}^{r} (t^{d_i} - 1)/(t-1)^2 \in \mathbb{F}_2[t]$$

where $2g+2 = d_1 + \cdots + d_r$ is a partition. We call a Weil polynomial *admissible* if it takes this form modulo 2. It is very simple to test whether a polynomial is admissible by explicitly trying all possible partitions of $2g+2$, and an isogeny class is guaranteed to not contain a hyperelliptic jacobian if its Weil polynomial is inadmissible. Thus admissibility is a necessary condition for a polynomial to be the Weil polynomial of a hyperelliptic curve.

By using asymptotic results on the number of Weil polynomials whose coefficients lie in prescribed congruence classes modulo an integer, we determine the asymptotic proportion of isogeny classes with admissible, or inadmissible, Weil polynomials.

Theorem 3.2 ([2], Theorem 3.3)**.** *Let $c(q,g)$ be the proportion of isogeny classes of g-dimensional abelian varieties over $\mathbb{F}_q$ with admissible Weil polynomial. For all $g \geq 2$ we have*

$$\lim_{q\to\infty} c(q,g) = \frac{Q(2g+2)}{2^g},$$

as q ranges over odd prime powers, where $Q(2g+2)$ is the number of partitions of $2g+2$ into distinct parts.

As the fraction $\frac{Q(2g+2)}{2^g}$ quickly converges to 0 as $g \to \infty$, we see that our necessary condition of admissibility rules out increasingly more isogeny classes of abelian varieties from containing a hyperelliptic jacobian as q and g go to infinity.

4 Vanishing of L-functions at the Central Point

A classical and difficult problem in analytic number theory is Chowla's conjecture [1] which predicts that the L-function attached to a primitive Dirichlet character does not vanish at $s = \frac{1}{2}$ (The value $s = \frac{1}{2}$ is called the *central point* due to it's location in the critical strip.) The simplest case of this conjecture concerns quadratic Dirichlet characters. Over function fields in positive characteristic, L-functions attached to quadratic Dirichlet characters are exactly the L-functions attached to hyperelliptic curves. In [9], Li showed that the analog of Chowla's conjecture in function fields does not hold and gave an explicit lower bound on the number of counter examples with bounded genus.

We extend this work by considering more general L-functions attached to Dirichlet characters of prime order. Our main strategy uses class field theory to connect an ℓ-th order Dirichlet character over $\mathbb{F}_q(t)$ whose L-function vanishes at the central point $s = \frac{1}{2}$ to a superelliptic curve which admits $\sqrt{q}$ as an eigenvalue for the Frobenius action on its jacobian. Moreover, all superelliptic curves which admit a dominant map to such a curve induce a character whose L-function also vanishes at $s = \frac{1}{2}$. Thus, a large part of our work is to prove a lower bound on the number of curves in hyperelliptic families which admit dominant maps to a fixed curve. We now set some notation and state our results:

$$\mathcal{A}_\ell(n, q) = \{\text{Primitive Dirichlet characters over } \mathbb{F}_q[t] \text{ of order } \ell \text{ with } \deg(c(\chi)) \leq n\},$$
$$\mathcal{B}_\ell(n, q) = \{\chi \in \mathcal{A}_\ell(n) \text{ such that } L(1/2, \chi) = 0\}$$

Theorem 4.1 ([3], Theorem 4.1)**.** *Let $\mathbb{F}_q$ be a finite field of odd characteristic p where $p \equiv 2 \bmod 3$, $q = p^{4e}$. Then for any $\epsilon > 0$, there exist positive constants C_ϵ and N_ϵ, such that $|\mathcal{B}_3(n, q)| \geq C_\epsilon \cdot q^{\frac{2n}{3} - \epsilon}$ for any $n > N_\epsilon$.*

For comparison, the set $\mathcal{A}_3(n)$ has cardinality $\sim Cnq^n$ for a positive constant C.
Finally, we are able to exploit the existence [10] of ℓ-th order superelliptic curves over $\overline{\mathbb{F}}_p$ that are supersingular to deduce the vanishing of ℓ-th order characters at the central point. The relevant result here is that a curve over $\mathbb{F}_q$ is supersingular if and only if each of the roots of it's zeta function is the product of $\sqrt{q}$ and a root of unity. Letting d be the order of the root of unity, we find that the base-changed curve C_{q^d} has $\sqrt{q}$ as it's Frobenius eigenvalue. Thus we obtain a result on the vanishing of L-functions of ℓ-th order Dirichlet characters over some finite fields.

Theorem 4.2 ([3], Theorem 4.2)**.** *Let $\ell > 2$ be a prime and $p \equiv -1 \bmod \ell$ a prime number. Then there exist positive integers d, a, B (all depending on ℓ and p) such that $|\mathcal{B}_\ell(n, q^d)| \geq Bq^{an}$ for all n sufficiently large.*

5 Newton Polygons and Ekedahl-Oort Types of Families of Curves

Given a geometrically integral, smooth projective curve C over $\mathbb{F}_q$ (a *nice curve*), denote it's jacobian by $\mathrm{Jac}(C)$. If C has genus g, then $\mathrm{Jac}(C)$ is an abelian variety of dimension g. When the curve C is specified, $\mathrm{Jac}(C)$ always has a canonical principal polarization.

The moduli space of principally polarized abelian varieties of a fixed dimension g, denoted $\mathcal{A}_g$, is a central object of study in arithmetic geometry. For $g \geq 2$, it has dimension $\frac{g(g+1)}{2}$. The moduli space of smooth curves of genus g, denoted $\mathfrak{M}_g$, is $3g - 3$ dimensional for $g \geq 2$. The jacobian construction yields a map $J : \mathfrak{M}_g \to \mathcal{A}_g$. For $g \geq 3$, this map is not surjective purely for dimension reasons. It is a classical problem to characterize the image of $\mathfrak{M}_g$ inside $\mathcal{A}_g$. A common approach to this problem in characterstic p involves stratifications of $\mathcal{A}_g$. A *stratification* is simply a decomposition of the space into a finite number of disjoint sets (called *strata*) that are locally closed. Even though the map J is typically not surjective, we can still ask if it's image intersects every stratum of a given stratification.

In characteristic $p > 0$, the space $\mathcal{A}_g$ admits many stratifications through which it is traditionally studied. The most common of these are the p-rank stratification, the Ekedahl-Oort stratification and the Newton stratification. On geometric points, a stratification can be expressed as a map to a finite set such that the fibers of this map yield the different strata. The p-rank stratification assigns to each $A \in \mathcal{A}_g$ an integer i

in the range $0 \leq i \leq g$ such that the number of p-torsion points in $A(\overline{\mathbb{F}}_p)$ is p^i. The integer i is called the *p-rank* of A. The possible isomorphism classes of $A[p]$, the p-torsion group scheme of A, is a finite set E_g for abelian varieties of a fixed dimension g. The assignment $A \mapsto A[p]$ thus yields a discrete invariant called the *Ekedahl-Oort type*. Finally, there are only finitely many possible isogeny classes of the p-divisible group $A[p^\infty]$. The set of these possibilities is denoted $\mathcal{N}_g$ and it's elements are called *Newton polygons*.

Both the E_g and $\mathcal{N}_g$ have natural combinatorial interpretations as directed posets with maximum and minimum elements. The set of abelian varieties in the maximum Ekedahl-Oort stratum is the same as those in the maximum Newton stratum. These are called *ordinary* abelian varieties. The abelian varieties corresponding to the minimum Newton polygon stratum are called *supersingular*. These abelian varieties are isogenous to a product of elliptic curves. The abelian varieties corresponding to the minimum Ekedahl-Oort stratum are called *superspecial*. These abelian varieties are isomorphic to a product of elliptic curves. Roughly speaking, the higher the stratum in these directed posets, the higher it's dimension is inside $\mathcal{A}_g$. For example, the ordinary stratum is always open and dense in $\mathcal{A}_g$ while the superspecial stratum always has dimension 0.

For abelian varieties of dimension $g = 1$, namely elliptic curves, there are only two possible Newton polygons (equivalently Ekedahl-Oort types) corresponding to ordinary and supersingular elliptic curves respectively. For slightly larger genera, i.e. $g = 2, 3$, for each possible Newton polygon or Ekedahl-Oort type, smooth curves have been found in every characteristic $p > 0$ with these invariants. For $p = 2$, van der Geer and van der Vlugt have shown that for every genus $g \geq 1$, there exists a supersingular curve of genus g over $\mathbb{F}_2$. For higher genera, the situation becomes difficult very quickly.

The groundbreaking work [10] of Li–Mantovan–Pries–Tang combines results of Moonen [11] and Viehmann-Wedhorn [13] to produce many hitherto-unknown Newton polygons and Ekedahl-Oort types arising from jacobians of curves. More precisely, they use twenty Shimura varieties described by Moonen and compute the exact Newton polygons and Ekedahl-Oort types that appear in each family using a result of Viehmann-Wedhorn. Moonen's original work implies that these twenty families are the only positive dimensional Shimura varieties that both arise from cyclic covers of $\mathbb{P}^1$ and are entirely contained in the open Torelli locus. So this method is exhausted for families of cyclic covers of $\mathbb{P}^1$.

We apply the same strategy to some PEL-type Shimura varieties arising from *non-abelian* Galois families of curves lying in the Torelli locus. In [4], the authors find twenty families of non-cyclic Galois covers of $\mathbb{P}^1$ whose images under the Torelli morphism are Shimura varieties. There are two significant points of departure from the cyclic case. First, since our curves are non-abelian, the structure of the set of possible Newton polygons realized using these families is more complicated. Second, the Shimura varieties attached to cyclic families are always unitary, whereas our non-abelian families yield some Shimura varieties coming from PEL moduli problems of type D to which the results of [13] do not apply. In my thesis, we generalize the method of [10] using the Chevalley-Weil formula to calculate the Newton polygons and Ekedahl-Oort types arising from the non-abelian families. We describe one such family of genus 4 curves in detail here.

This family consists of branched Galois covers of the projective line with Galois group Q_8 whose jacobians are dense in an irreducible component of a PEL moduli space of type D (in the sense of Kottwitz [7]). Using the fact that these curves are hyperelliptic, we are able use work of Shaska [12] to find an explicit equation defining this family of curves.

Consider the family of genus 4 curves defined over the finite field $\mathbb{F}_p$ given by the equation

$$C_t : y^2 = g_t(x) \text{ where } g_t(x) = x(x^4 - 1)(x^4 + tx^2 + 1).$$

This family yields a singular curve exactly when $t = 2, -2$ and we exclude always these fibers from our consideration. With the standard notation for Newton polygons as in [10], I have proven the following results in my thesis:

Theorem 5.1. *If $p > 7$, the family C_t has the following properties:*

1. *The only possible Newton polygons of the family C_t over $\overline{\mathbb{F}}_p$ are*

$$\begin{cases} \{ord^4, ss^4\} & \text{if } p \equiv 1, 7 \pmod 8 \\ \{ord^2 \oplus ss^2, ss^4\} & \text{if } p \equiv 3, 5 \pmod 8. \end{cases}$$

2. *For $p \equiv 1, 7 \pmod 8$, and $a \in \overline{\mathbb{F}}_p$, all but finitely many of the curves C_a have Newton polygon ord^4, i.e. are ordinary.*

3. *For $p \equiv 1, 7 \pmod 8$, there is always a $t_0 \in \overline{\mathbb{F}}_p \setminus \{-2, 2\}$ such that C_{t_0} is supersingular and superspecial.*

4. *For $p \equiv 1, 7 \pmod 8$ and $a \in \overline{\mathbb{F}}_p \setminus \{-2, 2\}$, a curve C_a is supersingular if and only if it is superspecial.*

5. *For $p \equiv 1, 7 \pmod 8$ and $a \in \overline{\mathbb{F}}_p \setminus \{-2, 2\}$, if C_a is supersingular, then $a \in \mathbb{F}_{p^2}$.*

6. *For $p \equiv 3, 5 \pmod 8$, and $a \in \overline{\mathbb{F}}_p$, all but finitely many of the curves C_a have Newton polygon ord$^2 \oplus ss^2$*

7. *For $p \equiv 3, 5 \pmod 8$, the curve C_0 is supersingular but not superspecial.*

The above proposition explicitly produces a supersingular curve of genus 4 in every characteristic greater than 7. This is a result that was first proven in [8] where the curves are produced as desingularizations of fiber products. However their elegant construction does not specify which subfield of $\overline{\mathbb{F}}_p$ these curves are defined over, nor give the defining equations of the family. By contrast, the structure of our family of curves shows that these curves are defined over $\overline{\mathbb{F}}_p$, the explicit equation of the curve as well as that it is hyperelliptic.

In addition, I am working on proving the following more precise conjectures in my thesis using the Picard-Fuchs differential equations attached to the curve family C_t and their hypergeometric solutions:

Conjecture 5.2. *Let $p \geq 7$ be a prime number and C_t be the genus 4 family of hyperelliptic curves defined above.*

1. *For $p \equiv 1 \pmod 8$, there are exactly $\frac{p-1}{8}$ superspecial curves in the family C_t.*

2. *For $p \equiv 1 \pmod{16}$, there is an $t_0 \in \mathbb{F}_p$ such that C_{t_0} is a supersingular curve if and only if p can be represented by both the quadratic forms $x^2 + 32y^2$ and $x^2 + 64y^2$.*

3. *For $p \equiv 7 \pmod 8$, there are exactly $\frac{p-7}{8}$ superspecial curves in the family C_t.*

6 Future Work

There are some natural extensions of our completed projects:

- Compute restrictions modulo n on Weil polynomials of *superelliptic curves*, i.e. curves whose function field is given by the equation $y^n = f(x)$ where $n \geq 2$ and $f(x)$ is a polynomial. Our work in [2] has focused on the case when $n = 2$ which yields hyperelliptic curves. A deeper understanding of the structure of the n-torsion of the Jacobians of superelliptic curves can generalize these results to $n > 2$. This is a project we intend to pursue directly after writing our thesis.

- Much of the machinery I have developed and code I have written as part of my thesis to systematically study the Newton polygons of Galois covers of $\mathbb{P}^1$ over a field of characteristic p can be expanded into an algorithm that takes in an *arbitrary* family of Galois covers of a given curve and returns all possible Newton Polygons and EO-types that appear in this family explicitly based solely on congruence conditions on the prime p. Since much of the technology needed for this extension can be "black-boxed", this project can be fruitfully segmented to provide for several undergraduate projects with students familiar with nothing more than basic abstract algebra.

- While our work has focused on the relationship between the moduli spaces of curves and abelian varieties via the jacobian construction, Prym varieties arising from double covers of curves also are an abundant source of abelian varieties. Recent work of Frediani-Groselli [5] has found that there are hundreds of positive dimensional special subvarieties inside the Prym locus of abelian varieties. We plan on computing the Newton Polygons and Ekedahl-Oort types associated to these families and extend our algorithms to this case.

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